



TITLE:

\mathbb{R}^N 上の楕円型問題の解について(非線形解析学と凸解析学の研究)

AUTHOR(S):

平野, 載倫

CITATION:

平野, 載倫. \mathbb{R}^N 上の楕円型問題の解について(非線形解析学と凸解析学の研究). 数理解析研究所講究録 1998, 1031: 101-109

ISSUE DATE:

1998-04

URL:

<http://hdl.handle.net/2433/61854>

RIGHT:

R^N 上の楕円型問題の解について

平野 載倫 (Yokohama National University)

1. Introduction . この講演では、次の楕円型問題の正值解の存在を考える。

$$(P) \quad \begin{cases} -\Delta u + u = g(x, u), & u > 0, \quad \text{in } R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}$$

where $f : R^N \rightarrow R$ and $g : \Omega \times R \rightarrow R$ is continuous with $g(x, 0) = 0$ for $x \in \Omega$. 楕円型問題 (P) については、過去 10 年間の間に、その解の存在と性質について多くの研究がなされている。最近、半線形楕円型問題

$$(P_Q) \quad \begin{cases} -\Delta u + u = Q(x) |u|^{p-1} u, & x \in R^N \\ u \in H^1(R^N), & N \geq 2 \end{cases}$$

の正值解については、 $1 < p$ for $N = 2$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$, および、 $Q(x)$ が positive bounded continuous function という条件下で、何人かの研究者によって、結果が得られている。 $Q(x)$ が radial function である場合には、問題 (P_Q) は無限個の解を持つことが得られる。これは、解を radial functions の中から探すことにより、常微分方程式に帰着させることによって得ることができる。(cf. [1]). $Q(x)$ が必ずしも radial でない場合には、領域が無限であることにより、compact 性の欠如という問題に突き当たる。すなわち、Sobolev type の compact embedding が成り立たない為に、解の存在が容易に示せないということになる。この問題は、P.L. Lions(cf.[6,7]) によって、いわゆる concentrate compactness method という方法によって部分的に解決された。この方法によれば、 P_Q のような問題は、適当な条件下で解くことができる。

P.L.Lions は彼の方法を用いて次のような結果を得た: Assume that

$$\lim_{|x| \rightarrow \infty} Q(x) = \bar{Q}(> 0) \quad \text{and} \quad Q(x) \geq \bar{Q} \quad \text{on } R^N,$$

then problem (P_Q) has a positive solution.

この結果は次のような観察に基づいている。すなわち、問題 P_Q の ground state level c_Q , すなわち

$$I_Q(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p+1} \int_{R^N} Q(x) u^{p+1} dx$$

の lowest critical level が ground state level $c_{\bar{Q}}$, すなわち $I_{\bar{Q}}$ の lowest critical level よりも小さい。このようなときには、我々は concentrate compactness method を用いること

ができる。問題 (P) についても同様の議論ができる。すなわち、 $g: R^N \times R \rightarrow R$ が条件 $\lim_{|x| \rightarrow \infty} g(x, t) = t^p$ を満たし、

$$I(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \int_{R^N} \int_0^{u(x)} g(x, t) dt dx,$$

$u \in H^1(R^N)$, の least critical level c_1 が

$$I^\infty(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + |u|^2) dx - \frac{1}{p+1} \int_{R^N} u^{p+1} dx.$$

のそれよりも小さいとする。このとき、適当な条件下で、(P) の正值解の存在は Ding & Ni[4] や Stuart[10] 等によって示されている。また、最近 Cao[2] は (P_Q) の正值解について、 $c_Q \leq c_{\bar{Q}}$ を満たす場合について、 $\lim_{\|x\| \rightarrow \infty} Q(x) = \bar{Q}$ かつ $Q(x) \geq 2^{(1-p)/2} \bar{Q}$ on R^N という条件下で証明している。

$c_Q = c_{\bar{Q}}$ が成り立つ場合には、concentrate compactness method を用いることができないので、証明が難しい。一方、 g が $Q(x)t^p$ という形で与えられていない場合は、Lagrange's method が使えない為に、新たな困難が生じる。 (P_Q) に関しては、解を得る為には、minimizing problem

$$\inf \{I_Q(u) : u \in V_\lambda\},$$

$$V_\lambda = \{u \in H^1(R^N), u > 0, \int_{R^N} Q(x) u^{p+1} dx = 1\}$$

の解を求めればよかった。すなわち、得られた解 u に対して c を適当に選べば、 cu が (P_Q) の解となる。Lagrange's method は残念ながら一般の g にたいしては有効に働かない。我々の方法は、こうした一般の場合にも有効となるものである。すなわち、 g が $g(0) = 0, g(t) \rightarrow t^p$ as $t \rightarrow \infty$ を満たす場合に問題 (P) の解を考えることができる。また、nonhomogeneous な場合:

$$(P_f) \quad \begin{cases} -\Delta u + u = |u|^{p-1} u + f, & x \in R^N \\ u \in H^1(R^N), & N \geq 3 \end{cases}$$

ここで $p > 1$ for $N = 1, 1 < p < (N+2)/(N-2)$ for $N \geq 3$, も同様の文脈で考えることができる。nonhomogeneous な場合については、Zhu[12] が解の存在を考えている。[12] において少なくとも 2 つの正值な (P) の解が次の条件下で存在することが示されている。すなわち、 $f \in L^2(R^N)$ は L^2 -norm が十分に小さく、exponential decay

$$f(x) \leq C \exp\{-(1+\epsilon)|x|\}, \quad \text{for } x \in R^N.$$

を持つ。我々の結果は、 $f \in L^q(R^N)$ ($q = (p+1)/p$) が正值であれば、その decay の速度については条件がいらぬ。

この講演では、次のような結果に対するアプローチを示す。

以下では、 $\|\cdot\|_q$ は $L^q(R^N)$ のノルムをあらわす。 $g: R^N \times R \rightarrow R$ にたいしては次のような条件を仮定する。

- (g1) There exists a positive number $d < 1$ such that

$$-dt + (1-d)t^p \leq g(x, t) \leq dt + (1+d)t^p$$
for all $(x, t) \in R^N \times [0, \infty)$;
- (g2) there exists a positive number C such that

$$|g_t(x, 0)| < 1 \text{ and } 0 < t^2 g_{tt}(x, t) < C(1+t^p)$$
for all $(x, t) \in R^N \times [0, \infty)$;
- (g3) $\lim_{|x| \rightarrow \infty} g(x, t) = |t|^{p-1} t$
uniformly on bounded intervals in $[0, \infty)$,

where $1 < p$ for $N = 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, and $g_t(\cdot)$ stands for the derivative of g with respect to the second variable.

この講演では次のような結果について述べる。

Theorem 1. (g2) および (g3) を仮定する。このとき、 $d_0 > 0$ が存在して、もし (g1) が $d < d_0$ なる値に対して成立するならば (P) は正值解を持つ。

(P_f), に関しては次の結果が成り立つ。

Theorem 2. $C > 0$ について 各 $f \in L^q(R^N)$ が $f \geq 0$ かつ $\|f\|_q < C$ を満たすならば、(P_f) は少なくとも二つの解を持つ。

以下では、上記の定理の証明の概略を与える。

2. Preliminaries. We just give a sketch of a proof of Theorem 1 to show that how the singular homology theory works for the proof of existence of positive solutions. We put $H = H^1(R^N)$. Then H is a Hilbert space with norm

$$\|u\| = \left(\int_{R^N} (|\nabla u|^2 + |u|^2) dx \right)^{1/2}.$$

The norm of the dual space $H^{-1}(R^N)$ of H is also denoted by $\|\cdot\|$. B_r stands for the open ball centered at 0 with radius r . We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^1(R^N)$ and $H^{-1}(R^N)$. For each $r > 1$, the norm of $L^r(R^N)$ is denoted by $\|\cdot\|_r$. For simplicity, we write $\|\cdot\|_*$ instead of $\|\cdot\|_{p+1}$. For $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$. We denote by C_p the minimal constant satisfying

$$\|u\|_* \leq C_p \|u\| \quad \text{for } u \in H. \quad (2.1)$$

It is easy to check that critical points of I are solutions of (P). It is also obvious that nonzero critical points of I^∞ are solutions of (P) with $g(t) = t^p$ for $t \geq 0$. For each functional F on H and $a \in R$, we set $F_a = \{u \in H : F(u) \leq a\}$. We put

$$M = \{u \in H \setminus \{0\} : \|u\|^2 = \int_{R^N} u g(x, u) dx\}$$

$$M^\infty = \{u \in H \setminus \{0\} : \|u\|^2 = \int_{R^N} u^{p+1} dx\}$$

For the proof of the following two propositions are crucial:

Proposition 2.1. *There exists positive number $d_0 < \tilde{d}_0$ and ϵ_0 satisfying that if (g1) holds with $d \leq d_0$, then for each $0 < \epsilon < \epsilon_0$,*

$$H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) = H_*(I_{c+\epsilon}, I_\epsilon)$$

where $H_*(A, B)$ denotes the singular homology group for a pair (A, B) of topological spaces (cf. Spanier[8]).

Proposition 2.2. *For each positive number $\epsilon < \epsilon_0$,*

$$H_q(I_{c+\epsilon}^\infty, I_\epsilon^\infty) = \begin{cases} 2 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Here we give a proof for Proposition 2.2.

We set

$$T_{u_\infty}(M^\infty) = \{\lim_{t \rightarrow 0} (c(t) - u_\infty)/t : c \in C^1((-1, 1); M^\infty) \text{ with } c(0) = u_\infty\},$$

$$\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+ = \{-\tau_x u_\infty : x \in R^N\} \cup \{\tau_x u_\infty : x \in R^N\}$$

and

$$T_{u_\infty}(\mathcal{C}) = \{\lim_{t \rightarrow 0} (u_\infty(\cdot + tx) - u_\infty(\cdot))/t : x \in R^N\}.$$

It follows from the definition of M^∞ that the codimension of $T_{u_\infty}(M^\infty)$ in H is one. It is also obvious that $\dim T_{u_\infty}(\mathcal{C}) = N$. We denote by \tilde{H} the subspace such that $H = \tilde{H} \oplus T_{u_\infty}(\mathcal{C})$. For each $r > 0$, we set $B_r^0 = B_r \cap \tilde{H}$. Here we consider the linealized equation

$$(L) \quad -\Delta u + u - h(x)u = \mu u, \quad u \in H, \mu \in R,$$

where $h(x) = p |u_\infty(x)|^{p-1}$ for $x \in R^N$. Since $-\Delta$ is positive definite and $h(x)I$ is compact, we find by Freidrich's theory that the negative spectrums of $A = -\Delta - h(x)I$ are finite and each eigenspace corresponding to a negative eigenvalue is finite dimensional. Then each eigenspace corresponding to a nonpositive eigenvalue of $L = -\Delta + I - h(x)I$ is finite dimensional. Then there exists $c_0 > 0$ and a decomposition $H = H_- \oplus H_0 \oplus H_+$ such that $H_0 = \ker(L)$ and L is positive(negative) definite on $H_+(H_-)$ with

$$\langle Lv, v \rangle \geq c_0 \|v\|^2 \quad (\leq -c_0 \|v\|^2) \quad \text{for } v \in H_+(H_-).$$

Since each $u \in \mathcal{C}$ is a solution of problem (P_∞) , we can see that $T_{u_\infty}(\mathcal{C}) \subset H_0$.

Lemma 2.3. $\dim H_- = 1$.

Proof. Since I^∞ attains its minimal on M^∞ at u_∞ , we have that $T_{u_\infty}(M^\infty) \subset H_+ \oplus H_0$. Then since the codimension of M^∞ is one, we find that $\dim H_- \leq 1$. On the other hand, we have

$$\begin{aligned} \langle Lu_\infty, u_\infty \rangle &= \int_{R^N} (|\nabla u_\infty|^2 + |u_\infty|^2 - p |u_\infty|^{p+1}) dx \\ &< \int_{R^N} (|\nabla u_\infty|^2 + |u_\infty|^2 - |u_\infty|^{p+1}) dx = 0. \end{aligned} \quad (2.2)$$

Then we have that $\dim H_- \geq 1$. This completes the proof. ■

In the following we denote by φ an element of H_- with $\|\varphi\| = 1$. Here we note that since $h \in C^\infty(R^N)$, each solution u of (L) is in $C^1(R^N)$. It then follows that if u has the form

$$u(r, \theta) = \psi(r)\xi(\theta_1, \dots, \theta_{n-1}), \quad \text{with } \xi \neq \text{const.},$$

in spherical coordinate, ψ satisfies that $\psi(0) = 0$.

We denote by H_r the set of all radial functions in H and by (L_r) the problem (L) restricted to H_r . Then, in spherical coordinates, the problem (L_r) with $\mu > 0$ is reduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + (h-1)\psi = -\mu\psi(r), \quad r > 0, \psi \in C_r, \quad (2.3)$$

$$\frac{d\psi(r)}{dr}(0) = 0, \quad (2.4)$$

where $C_r = \{\psi \in C[0, \infty) : \lim_{r \rightarrow \infty} \psi(r) = 0\}$.

We next consider nonradial solutions of (L). In case of nonradial functions, the problem (L) is deduced to

$$\psi''(r) + \frac{n-1}{r}\psi'(r) + ((h-1) - \frac{\alpha_k}{r^2})\psi(r) = -\mu\psi(r), \quad r > 0, \quad (2.5)$$

$$\psi(0) = 0 \quad (2.6)$$

where $\psi \in H_r$, $\alpha_k = k(k+n-1)$, $k = 1, 2, \dots$. Note that α_k are the eigenvalues of Laplacian $-\Delta$ on S^{n-1} , the unit sphere, and the dimension of the eigenspace S_k associate with α_k is

$$\rho_k = \binom{k+n-2}{k} \frac{n+2k-2}{n+k-2}.$$

That is there exists smooth functions $\{\varphi_{k,i} : i = 1, \dots, \rho_k\}$ defined on S^{n-1} such that $S_k = \text{span}\{\varphi_{k,1}, \dots, \varphi_{k,\rho_k}\}$, and the functions $u = \psi(r)\varphi_{k,i}(\theta)$ are the solutions of (L).

By using (2.5) and (2.6), we can see

Lemma 2.4. $\dim H_0 \leq N + 1$.

Here we recall that H has a decomposition $H = \tilde{H} \oplus T_{u_\infty}(\mathcal{C})$ and then $H = \tau_x \tilde{H} \oplus \tau_x T_{u_\infty}(\mathcal{C})$ for each $x \in R^N$. Then since \mathcal{C}_\pm are smooth N -manifolds, we have that there exists $r_0 > 0$ such that

$$\tau_x((-1)^i u_\infty + B_{r_0}^0) \cap \tau_y(u_\infty + B_{r_0}^0) = \emptyset \quad (2.7)$$

for all $x, y \in R^N$ with $x \neq y$, and $i = 0, 1$. Here we consider a restriction $I^\infty|_{u_\infty + \tilde{H}}$ of I^∞ on $u_\infty + \tilde{H}$. Then from Lemma 3.2 and Lemma 3.3, we have by Gromoll-Meyer theory[3] that there exists subspaces $H_1, H_{2,1}, H_{2,2}$ of \tilde{H} , a positive number $r_1 < r_0$, a mapping $\beta \in C^1((H_{2,2} \cap B_{r_1}^0), R)$ and a homeomorphism $\psi : u_\infty + B_{r_1}^0 \rightarrow u_\infty + \tilde{H}$ such that $\tilde{H} = H_1 \oplus H_{2,1} \oplus H_{2,2}$ and

$$I^\infty|_{u_\infty + \tilde{H}}(\psi(u)) = c - \|u_1\|^2 + \|u_{2,1}\|^2 + \beta(u_{2,2}) \quad (2.8)$$

for each $u \in u_\infty + B_{r_1}^0$ with $u = u_\infty + u_1 + u_{2,1} + u_{2,2}$, $u_1 \in H_1$, $u_{2,i} \in H_{2,i}$, $i = 1, 2$. It follows from Lemma 2.3 that $H_{2,2}$ is one dimensional. Noting that $T_{u_\infty}(M) \subset H_0 \oplus H_+$ and u_∞ is the minimal point of I^∞ on M , we have by choosing r_1 sufficiently small that $\beta(t\varphi_2)$ is strictly increasing as $|t|$ increases in $[-r_1, r_1]$, where $\varphi_2 \in H_{2,2}$ with $\|\varphi_2\| = 1$.

Since I^∞ is even, it is obvious that I^∞ has the form (2.8) on $-(u_\infty + B_{r_1}^0)$. We also note that for each $x \in R^N$, (2.8) holds for each $u \in \tau_x(u_\infty + B_{r_0}^0)$ with ψ replaced by $\tau_{-x} \circ \psi$.

Proof of Proposition 2.2. By the deformation property(cf. theorem 1.2 of Chang[3]) and the homotopy invariance of the homology groups, we have

$$H_q(I_{c+\epsilon}^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_{c-\epsilon}^\infty), \text{ and}$$

$$H_q(I_c^\infty \setminus \mathcal{C}, I_{c-\epsilon}^\infty) \cong H_q(I_{c-\epsilon}^\infty, I_{c-\epsilon}^\infty) \cong 0.$$

From the exactness of the singular homology groups ,

$$\begin{aligned} H_q(I_c^\infty \setminus \mathcal{C}, I_{c-\epsilon}^\infty) &\rightarrow H_q(I_c^\infty, I_{c-\epsilon}^\infty) \rightarrow H_q(I_c^\infty, I_c^\infty \setminus \mathcal{C}) \\ &\rightarrow H_{q-1}(I_c^\infty \setminus \mathcal{C}, I_{c-\epsilon}^\infty) \rightarrow \dots \end{aligned}$$

we find

$$0 \rightarrow H_q(I_c^\infty, I_{c-\epsilon}^\infty) \rightarrow H_q(I_c^\infty, I_c^\infty \setminus \mathcal{C}) \rightarrow 0.$$

That is

$$H_q(I_c^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_c^\infty \setminus \mathcal{C}).$$

Noting that $\cup\{\tau_x(\pm u_\infty + B_{r_1}^0) : x \in R^N\}$ are disjoint open neighborhoods of \mathcal{C}_\pm respectively, and that I_c^∞ is invariant under the translations τ_x , we find from the excision property and (2.8) that

$$\begin{aligned} H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) &\cong H_*(I_c^\infty, I_c^\infty \setminus \mathcal{C}) \\ &\cong H_*(I_c^\infty \cap (\cup_{i=\pm 1} \cup_x \tau_x(iu_\infty + B_{r_1}^0)), \\ &\quad I_c^\infty \cap (\cup_{i=\pm 1} \cup_x \tau_x(iu_\infty + B_{r_1}^0) \setminus \mathcal{C})) \\ &\cong H_*(u_\infty + B_{r_1}^1, (u_\infty + B_{r_1}^1) \setminus \{u_\infty\}) \\ &\quad \oplus H_*(-u_\infty + B_{r_1}^1, (-u_\infty + B_{r_1}^1) \setminus \{u_\infty\}) \\ &\cong H_*([0, 1], \{0, 1\}) \oplus H_*([0, 1], \{0, 1\}). \end{aligned}$$

This completes the proof. ■

3. Proof of Theorem 1. We next consider a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying the following conditions:

- (1) $U \cap (-U) = \emptyset$;
- (2) $\{\tau_x u_\infty : |x| \geq r\} \subset \text{int} K$ for some $r > 0$;
- (3) $cl(I_{c+\epsilon} \cap K) \subset \text{int}(I_{c+\epsilon} \cap U)$;
- (4) $H_{N-1}(I_{c+\epsilon} \cap U) = 1$, $H_1(I_{c+\epsilon} \cap U) = 0$;
- (5) I_ϵ is a strong deformation retract of $I_{c+\epsilon} \setminus (K \cup (-K))$;
- (6) $H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) = 2$ or $H_0((I_{c+\epsilon} \cap U) \setminus K) \geq 2$ holds.

Proposition 3.1. *There exists a triple $(U, K, \epsilon) \subset H \times H \times R^+$ which satisfies (1) - (6).*

We omit the proof of Proposition 3.1.

Lemma 3.2. *Suppose that there exist a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying (1)-(6). Suppose in addition that $H_{N-1}((I_{c+\epsilon} \cap U) \setminus K) \geq 2$. Then $H_N(I_{c+\epsilon}, I_\epsilon) \geq 2$.*

Proof. We put $\tilde{K} = K \cup (-K)$. Since I_ϵ is a strong deformation retract of $I_{c+\epsilon} \setminus \tilde{K}$, we find that

$$H_q(I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.$$

Then from the exactness of the singular homology groups of the triple $(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}, I_\epsilon)$ we have

$$0 \rightarrow H_q(I_{c+\epsilon}, I_\epsilon) \rightarrow H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \rightarrow 0.$$

That is

$$H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}).$$

From (1), we find

$$H_q(I_{c+\epsilon}, I_{c+\epsilon} \setminus \tilde{K}) \cong H_q(W, W \setminus K) \oplus H_q(-W, (-W) \setminus (-K))$$

where $W = I_{c+\epsilon} \cap U$. Then since $H_{N-1}(W \setminus K) \geq 2$, we have from (4) and the exactness of the sequence

$$\rightarrow H_q(W, W \setminus K) \rightarrow H_{q-1}(W \setminus K) \rightarrow H_{q-1}(W) \rightarrow H_{q-1}(W, W \setminus K) \rightarrow \quad (3.1)$$

with $q = N$ that $H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W \setminus K) \oplus H_N(W, W \setminus K) \geq 2$. ■

Lemma 3.3. Suppose that $(U, K, \epsilon) \subset H \times H \times R^+$ satisfies (1) - (6). Suppose in addition that $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U) \setminus K) = 1$. Then $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$ holds.

We can now prove Theorem 1.

Proof of Theorem. Let (U, K, ϵ) be the triple constructed above. We have by Proposition 2.1 and Proposition 2.2 that $H_1(I_{c+\epsilon}, I_\epsilon) = 2$ and $H_q(I_{c+\epsilon}, I_\epsilon) = 0$ for $q \neq 1$. Now suppose that $(I_{c+\epsilon} \cap U) \setminus K$ is disconnected. Then since $H_0((I_{c+\epsilon} \cap U) \setminus K) \geq 2$, we find by Lemma 3.2 that $H_N(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. On the other hand, if $U \setminus K$ is connected, then $H_0(U \setminus K) = 1$. Then by Lemma 3.3, we have $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. Thus we obtain that there exists a positive solution of (P). ■

REFERENCES

- [1] Berestycki H. & Lions P. L., Nonlinear scalar field equations, I, II, Archs ration Mech. Analysis 82(1982), 313-376.
- [2] Cao D-M, Positive solutions and bifurcation from the essential spectrum of a semilinear elliptic equations on R^N , Nonlinear Analysis TMA 15(1990), 1045-1052.
- [3] Chang K. C, indefinite dimensional Morse theory and its applications, Séminaire de Mathématiques Supérieures no 97, Univ. de Montreal(1985).
- [4] Ding W. Y. & Ni W. M., On the existence of positive solutions of a semilinear elliptic equation, Archs ration Mech. Analysis 91(1986), 283-307.
- [5] Kwong M. K., Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^N , Archs ration Mech. Analysis 105(1989), 243-266.

- [6] Lions P.L., The concentration-compactness principle in the calculus of variations, the locally compact case. Part I, Ann. Inst. H. Poincaré Analyses non Linéaire 1(1984), 109-145.
- [7] Lions P.L., The concentration-compactness principle in the calculus of variations, the locally compact case. Part II, Ann. Inst. H. Poincaré Analyses non Linéaire 1(1984), 223-283.
- [8] Spanier E, Algebraic Topology, McGraw-Hill, New York (1966).
- [9] Strauss W. A., Existence of solitary waves in higher dimensions, Commun. math. Phys. 55(1977), 149-162.
- [10] Stuart C. A., Bifurcation from the essential spectrum for some noncompact nonlinearities, to appear
- [11] Zhu, Xi-Ping, Multiple entire solutions of a semilinear elliptic equation, Nonlinear Analysis TMA 12(1988), 1297-1316.
- [12] Zhu, Xi-Ping, A perturbation result on positive entire solutions of a semilinear elliptic equations, J. Diff. Equ. 92(1991), 163-178.